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Group Report 1964-43

E. B. Temple

A Method for Evaluating a Generalized Fresnel Integral Related to the Spectra of Amplitude and Frequency Modulated Pulses

5 August 1964

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A METHOD FOR EVALUATING A GENERALIZED FRESNEL INTEGRAL RELATED TO THE SPECTRA OF AMPLITUDE AND FREQUENCY MODULATED PULSES

E. B. TEMPLE

Group 41

GROUP REPORT 1964-43

5 AUGUST 1964

LEXINGTON MASSACHUSETTS

Abstract

Asymptotic expressions for the Fourier spectra of pulses modulated by a carrier frequency plus linear FM are obtained for several pulse shapes, using an asymptotic series for the complex Fresuel integral. A geometric interpretation of the terms in this series is given.

Accepted for the Air Force
Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

A Method for Evaluating a Generalized Fresnel Integral
Related to the Spectra of Amplitude and Frequency Modulated Pulses

A. Statement of the Problem

Recently questions have arisen concerning the magnitude of the interference occurring between adjacent radars at the present AMRAD site. These questions have created the need for evaluating the integral

$$G(f, \alpha) = \int_{-\infty}^{\infty} g(t)\, e^{i\alpha t^2}\, e^{-i2\pi ft}\, dt \qquad (1)$$

for various functional forms of $g(t)$ and for various ranges of the magnitudes of the essential parameters fT and αT^2, where

t = time; f = frequency in cycles per second;*

α = the frequency modulation parameter in radians/sec^2;

T = the pulse duration in seconds; (2)

$g(t) = h(t)$ in $0 \leq t \leq T$; $h(t)$ = either a real or a complex function;

$g(t) = 0$ in $t < 0$, $t > T$.

When the function $g(t)$ is defined in the above manner, Eq. (1) can be rewritten as follows:

$$G(f, \alpha) = \int_{0}^{T} h(t)\, e^{i\left[\alpha t^2 - 2\pi ft\right]}\, dt \quad . \qquad (3)$$

---

*The variable f can be replaced, if the analysis required it, by the difference frequency $\Delta f = f - f_o$, where f_o is the CARRIER FREQUENCY and f is the frequency at which the spectrum $G(f)$ is viewed. In this case, f is required to lie in a range corresponding to the inequality stated in Eq. (6), but is otherwise arbitrary.

Evaluation of this integral yields in general the spectrum of a general pulse $h(t)$ modulated in amplitude, phase and frequency. The first two types of modulation correspond to variations in the amplitude and phase, respectively, of the complex function $h(t)$. Frequency modulation corresponds to non-vanishing values of the parameter α. In the present paper, only amplitude and frequency modulation will be considered, i. e., $h(t)$ will be assumed to be a real function of t.

The integral in Eq. (3) is a generalized Fresnel integral in the sense that for arbitrary functional forms of $h(t)$ it is a generalization of the integral

$$I(f, \alpha) = \int_0^T e^{i\left[\alpha t^2 - 2\pi f t\right]} dt \quad , \tag{4}$$

which can be reduced, with the aid of the proper change of variables, to a difference of two classical Fresnel integrals of the type encountered in Optics, namely, of the following type:

$$J(u) = \int_0^u e^{i\frac{\pi}{2}\xi^2} d\xi \quad . \tag{5}$$

Now the behavior of the special integrals, Eqs. (4) and (5), is well known, and extensive tables are available for use in evaluating them. The same cannot be said, however, for the general case, when $h(t)$ in Eq. (3) is other than a constant. Very little information is available in the literature regarding the behaviour of the generalized integral. In the absence of such information, it was necessary to devise an original approach to the evaluation of Eq. (3). The approach developed by the author and employed in the present paper is outlined in Section C.

B. Summary of Results

The integral in Eq. (3) has been evaluated for nine different cases -- eight of them specific cases, the other a general case. In each case, the essential parameters fT and αT^2 were assumed to obey the following inequality:

$$2\pi fT >> \alpha T^2 >> 1 , \tag{6}$$

or equivalently, in terms of variables Λ and A introduced later in the analysis, the following inequality;

$$\Lambda >> A >> \frac{1}{2\pi A} \ ; \ \Lambda = \frac{fT}{A} \ ; \ A^2 = \frac{\alpha T^2}{2\pi} \ . \tag{7}$$

The general case referred to above corresponds to the following representation of the function $h(t)$ in the interval $0 \leq t \leq T$: $h(t)$ a <u>piece-wise-linear function.</u>*

In this case, assuming the interval $0 \leq t \leq T$ to be divided into N equal intervals, each of length T/N, we have

$$h(t) = h_n(t) \ \text{in} \ (n-1)\left(\frac{T}{N}\right) < t < (n)\left(\frac{T}{N}\right) \ ; \tag{8}$$

where

$$h_n(t) = a_{on} + a_{1n} t \ . \tag{9}$$

The eight specific pulse shapes which were treated, together with the corresponding definitions of the function $h(t)$, are listed below and also in Fig. 1. In the equations below, the function $h(t)$ is followed in each case by the associated spectral function $G(f, \alpha)$ obtained by evaluating Eq. (3)

---

*A method for calculating the FM spectrum of a pulse approximated by a piecewise linear function is outlined in Section D.

with the aid of the methods outlined in Section C.*

1. Rectangular Pulse

$$h(t) = 1 \tag{10}$$

$$G(f,\alpha) \cong i\left(\frac{1}{2\pi}\right)\left[\frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f - \frac{\alpha T}{\pi}\right)} - \frac{1}{f}\right] + 0\left(\frac{1}{f^3}\right). \tag{11}$$

2. Triangular Pulse

$$\begin{aligned} h(t) &= \frac{2t}{T} \quad \text{in} \quad 0 \leq t \leq \frac{T}{2} \\ h(t) &= \left(\frac{2}{T}\right)(-t + T) \quad \text{in} \quad \frac{T}{2} < t \leq T \end{aligned} \tag{12}$$

$$G(f,\alpha) \cong -\left(\frac{1}{2\pi^2 T}\right)\left[\frac{1}{f^2} - \frac{2\exp\left[i\left\{\alpha T^2/4 - \pi f T\right\}\right]}{\left(f - \frac{\alpha T}{2\pi}\right)^2} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f - \frac{\alpha T}{\pi}\right)^2}\right] + 0\left(\frac{1}{f^4}\right). \tag{13}$$

---

*The expressions for $G(f,\alpha)$ given in Eqs. (11), (13), (15), (17), (19), (21), (23), and (25) were obtained in each case by approximating $G(f,\alpha)$ by the first non-vanishing term by the asymptotic expansion of $G(f,\alpha)$ (of the type derived in Section C). When $G(f,\alpha)$ consists of the sum of several separate expansions, taking the first term means adding the first term (non-vanishing) of each of the component expansions. For example, in Eq. (43) one would take the sum of the first non-vanishing terms of the three expansions $\sum_{m=o}^{M}$, $\sum_{m=1}^{M}$, and $\sum_{m=2}^{M}$.

3. Parabolic Pulse

$$h(t) = \left(\frac{4t}{T}\right)\left(1 - \frac{t}{T}\right) \tag{14}$$

$$G(f, \alpha) \cong -\left(\frac{1}{\pi^2 T}\right)\left[\frac{1}{f^2} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f - \frac{\alpha T}{\pi}\right)^2}\right]$$

$$+ i\left(\frac{1}{\pi^3 T^2}\right)\left[-\frac{1}{f^3} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f - \frac{\alpha T}{\pi}\right)^3}\right] + 0\left(\frac{1}{f^4}\right) \tag{15}$$

4. Trapezoidal Pulse

$$h(t) = h_1(t) = t/\tau_- \quad \text{in} \quad 0 < t < \tau_-; \quad \tau_- = \frac{T - \epsilon}{2}; \quad \epsilon \le T;$$

$$h(t) = h_2(t) = 1 \quad \text{in} \quad \tau_- < t < \tau_+; \quad \tau_+ = \frac{T + \epsilon}{2} \tag{16}$$

$$h(t) = h_3(t) = (-t + T)/\tau_- \quad \text{in} \quad \tau_+ < t < T$$

$$G(f, \alpha) \cong -\left(\frac{1}{4\pi^2 \tau_-}\right)\left[\frac{1}{f^2} - \frac{\exp\left[i\left\{\alpha \tau_-^2 - 2\pi f \tau_-\right\}\right]}{\left(f - \frac{\alpha \tau_-}{\pi}\right)^2}\right.$$

$$\left. - \frac{\exp\left[i\left\{\alpha \tau_+^2 - 2\pi f \tau_+\right\}\right]}{\left(f - \frac{\alpha \tau_+}{\pi}\right)^2} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f - \frac{\alpha T}{\pi}\right)^2}\right] + 0\left(\frac{1}{f^4}\right) \tag{17}$$

5. Parabolinear Pulse

$$\left.\begin{aligned}
&h(t) = h_1(t) = a_{11} t \quad \text{in} \quad 0 < t < \tau_-; \quad \tau_- = \frac{T - \epsilon}{2}; \\
&h(t) = h_2(t) = 1 - \left(\frac{a_{11}}{\epsilon}\right)\left(t - \frac{T}{2}\right)^2 \quad \text{in} \quad \tau_- < t < \tau_+; \\
&h(t) = h_3(t) = a_{11}(-t + T) \quad \text{in} \quad \tau_+ < t < T; \\
&\tau_- = \frac{T - \epsilon}{2}; \quad \tau_+ = \frac{T + \epsilon}{2}; \quad a_{11} = \frac{4}{2T - \epsilon} \equiv \frac{4}{4\tau_- + \epsilon}.
\end{aligned}\right\} \tag{18}$$

$$G(f,\alpha) \cong -(\frac{a_{11}}{4\pi^2}) \left[\frac{1}{f^2} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{(f - \frac{\alpha T}{\pi})^2} \right]$$

$$+ i(\frac{a_{11}}{4\pi^3 \epsilon}) \left[- \frac{\exp\left[i\left\{\alpha \tau_-^2 - 2\pi f \tau_-\right\}\right]}{(f - \frac{\alpha \tau_-}{\pi})^3} \right.$$

$$\left. + \frac{\exp\left[i\left\{\alpha \tau_+^2 - 2\pi f \tau_+\right\}\right]}{(f - \frac{\alpha \tau_+}{\pi})^3} \right] + 0(\frac{1}{f^4}) \qquad (19)$$

where

$$-(\frac{a_{11}}{4\pi^2}) = - (\frac{1}{\pi^2(2T - \epsilon)}) \; ; \; i(\frac{a_{11}}{4\pi^3 \epsilon}) = (\frac{i}{(\pi^3 \epsilon)(2T - \epsilon)}) \qquad (19a)$$

6. Sinusoidal Pulse

$$h(t) = \sin(\frac{\pi t}{T}) = (-\frac{i}{2}) \left[\exp i(\frac{\pi t}{T}) - \exp i(-\frac{\pi t}{T})\right] \qquad (20)$$

$$G(f,\alpha) \cong (\frac{1}{4\pi}) \left[\left\{ - \frac{1}{(f - \frac{1}{2T})} + \frac{1}{(f + \frac{1}{2T})} \right\} \right.$$

$$\left. + \exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right] \left\{ - \frac{1}{(f - \frac{\alpha T}{\pi}) - \frac{1}{2T}} + \frac{1}{(f - \frac{\alpha T}{\pi}) + \frac{1}{2T}} \right\} \right] + 0(\frac{1}{f^4})$$

$$\equiv (-\frac{1}{4\pi T}) \left[\frac{1}{f^2 - \frac{1}{4T^2}} + \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{(f - \frac{\alpha T}{\pi})^2 - \frac{1}{4T^2}} \right] + 0(\frac{1}{f^4}) \qquad (21)$$

7. Sine-Squared Pulse

$$h(t) = \sin^2\left(\frac{\pi t}{T}\right) \equiv -\frac{1}{4}\left(e^{i\frac{\pi t}{T}} - e^{-i\frac{\pi t}{T}}\right)^2 \tag{22}$$

$$G(f,\alpha) \cong \left(-\frac{i}{2\pi}\right)\left[\left(\frac{1}{2f} - \frac{1}{4\left(f-\frac{1}{T}\right)} - \frac{1}{4\left(f+\frac{1}{T}\right)}\right)\right.$$

$$\left. + \exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]\left(\frac{-1}{2\left(f-\frac{\alpha T}{\pi}\right)} + \frac{1}{4\left(f-\frac{\alpha T}{\pi}-\frac{1}{T}\right)} + \frac{1}{4\left(f-\frac{\alpha T}{\pi}+\frac{1}{T}\right)}\right)\right]$$

$$\equiv \left(\frac{i}{4\pi T^2}\right)\left[\frac{1}{f\left(f^2-\frac{1}{T^2}\right)} - \frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f-\frac{\alpha T}{\pi}\right)\left[\left(f-\frac{\alpha T}{\pi}\right)^2-\frac{1}{T^2}\right]}\right] + 0\left(\frac{1}{f^5}\right) \tag{23}$$

8. Sine-of-Θ-Squared Pulse

$$h(t) = \sin \Theta^2(t); \quad \Theta^2(t) = \pi\left[\frac{(T-t)^2}{T^2}\right] \tag{24}$$

$$G(f,\alpha) \cong \left(\frac{1}{4\pi}\right)\left[\frac{1}{\left(f+\frac{1}{T}\right)} - \frac{1}{\left(f-\frac{1}{T}\right)}\right] + \left(-\frac{i}{4\pi^2 T^2}\right)\left[\frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f-\frac{\alpha T}{\pi}\right)^3}\right]$$

$$+ \left(-\frac{i}{8\pi^3}\right)\left[\frac{\alpha + \frac{\pi}{T^2}}{\left(f+\frac{1}{T}\right)^3} - \frac{\alpha - \frac{\pi}{T^2}}{\left(f-\frac{1}{T}\right)^3}\right] + 0\left(\frac{1}{f^6}\right)$$

$$\equiv \left(-\frac{1}{2\pi T}\right)\left[\frac{1}{\left(f^2-\frac{1}{T^2}\right)}\right] + \left(-\frac{i}{4\pi^2 T^2}\right)\left[\frac{\exp\left[i\left\{\alpha T^2 - 2\pi f T\right\}\right]}{\left(f-\frac{\alpha T}{\pi}\right)^3}\right]$$

$$+ \left(-\frac{i}{4\pi^2 T^2}\right)\left[\frac{f}{\left(f^2-\frac{1}{T^2}\right)^2}\right] + \left(-\frac{\alpha i}{4\pi^3 T}\right)\left[\frac{1}{\left(f^2-\frac{1}{T^2}\right)^2}\right]$$

$$+ \left(\frac{\alpha i}{\pi^3 T}\right)\left[\frac{f^2}{\left(f^2-\frac{1}{T^2}\right)^3}\right] + \left(-\frac{i}{\pi^2 T^4}\right)\left[\frac{f}{\left(f^2-\frac{1}{T^2}\right)^3}\right] + 0\left(\frac{1}{f^6}\right). \tag{25}$$

The sine-of-Θ-squared pulse is a rather interesting one, since it resembles certain physically realizable pulses having rapid rise times and well-defined tails. Moreover, the leading edge (the part to the left of the peak) or the trailing edge, or both, could be fitted together with other functions to approximate other pulses having one or both of the properties mentioned above. Because of the practical potentialities of the sine-of-Θ-squared pulse, a detailed version of this pulse is presented in Fig. 2.

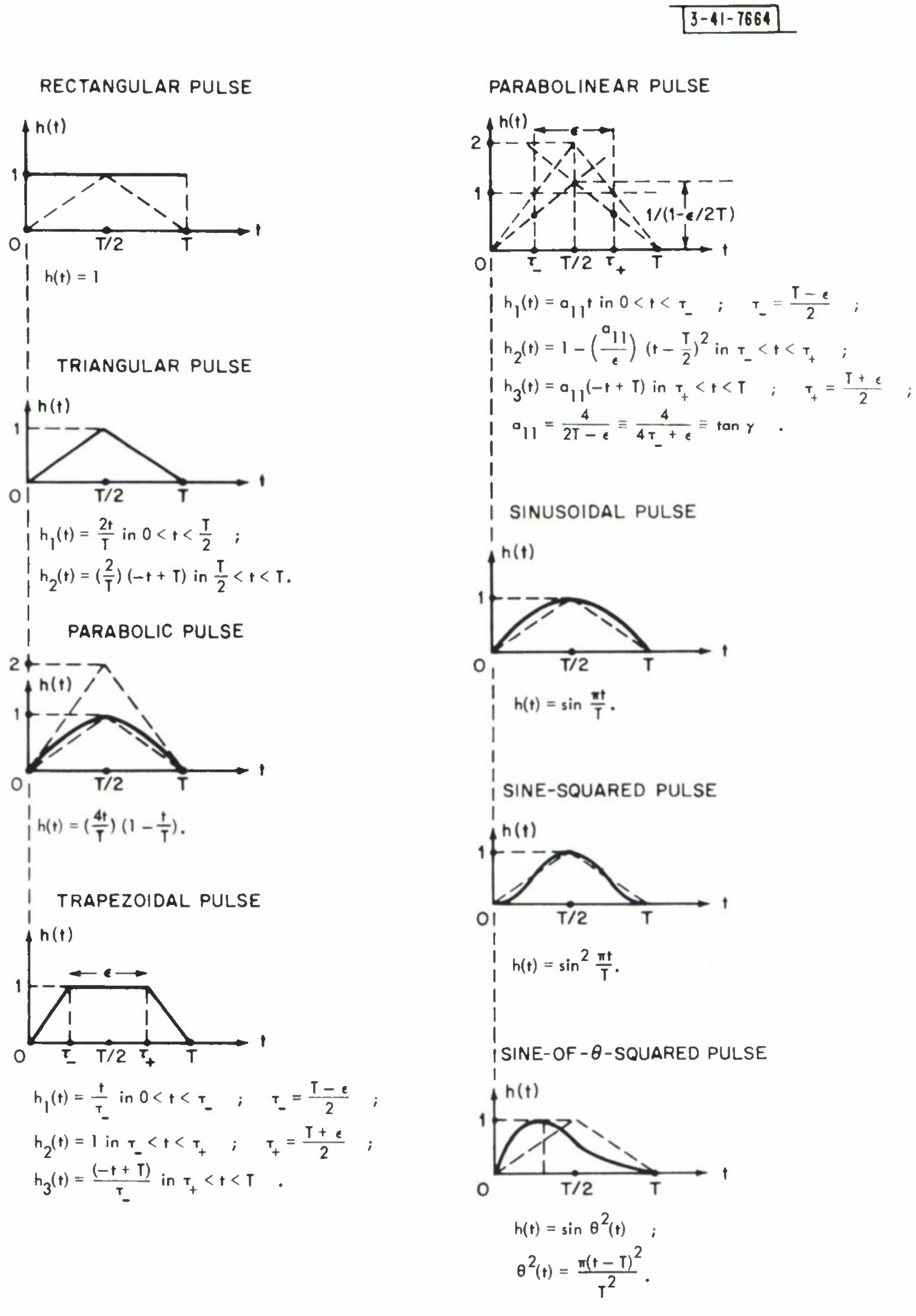
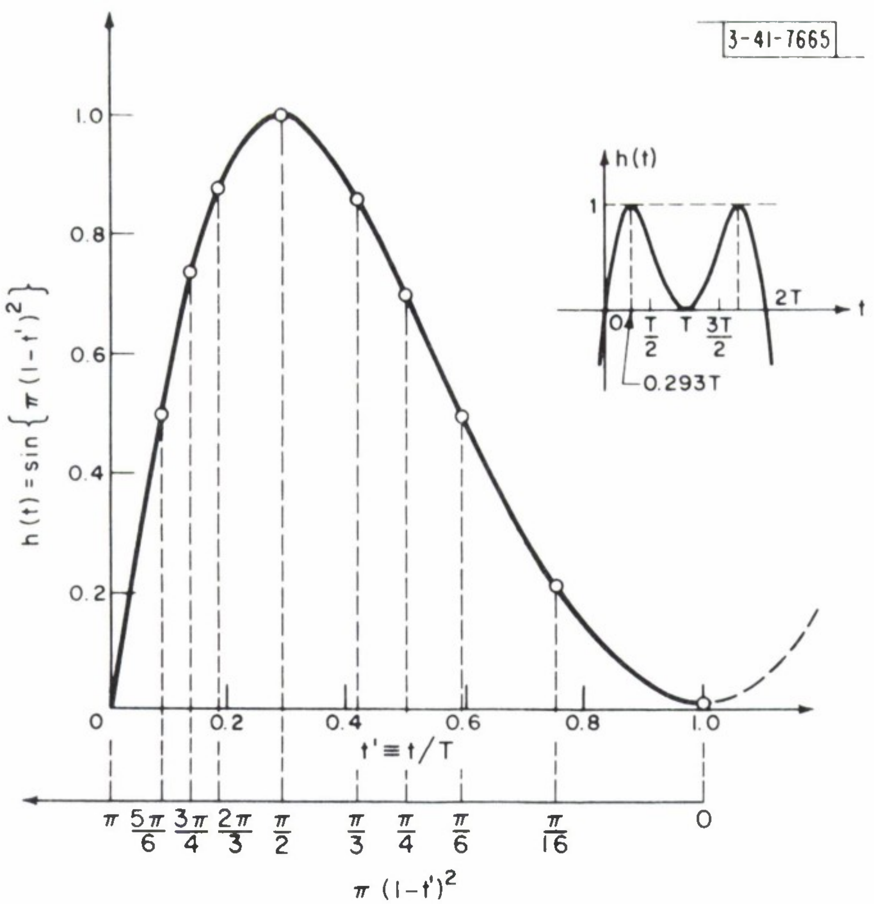

3-41-7664
RECTANGULAR PULSE
$h(t) = 1$
TRIANGULAR PULSE
$h_1(t) = \frac{2t}{T}$ in $0 < t < \frac{T}{2}$;
$h_2(t) = (\frac{2}{T})(-t + T)$ in $\frac{T}{2} < t < T$.
PARABOLIC PULSE
$h(t) = (\frac{4t}{T})(1 - \frac{t}{T})$.
TRAPEZOIDAL PULSE
$h_1(t) = \frac{t}{\tau_-}$ in $0 < t < \tau_-$; $\tau_- = \frac{T - \epsilon}{2}$;
$h_2(t) = 1$ in $\tau_- < t < \tau_+$; $\tau_+ = \frac{T + \epsilon}{2}$;
$h_3(t) = \frac{(-t + T)}{\tau_-}$ in $\tau_+ < t < T$.
PARABOLINEAR PULSE
$1/(1-\epsilon/2T)$
$h_1(t) = a_{11}t$ in $0 < t < \tau_-$; $\tau_- = \frac{T - \epsilon}{2}$;
$h_2(t) = 1 - \left(\frac{a_{11}}{\epsilon}\right)(t - \frac{T}{2})^2$ in $\tau_- < t < \tau_+$;
$h_3(t) = a_{11}(-t + T)$ in $\tau_+ < t < T$; $\tau_+ = \frac{T + \epsilon}{2}$;
$a_{11} = \frac{4}{2T - \epsilon} \equiv \frac{4}{4\tau_- + \epsilon} \equiv \tan\gamma$.
SINUSOIDAL PULSE
$h(t) = \sin\frac{\pi t}{T}$.
SINE-SQUARED PULSE
$h(t) = \sin^2\frac{\pi t}{T}$.
SINE-OF-θ-SQUARED PULSE
$h(t) = \sin\theta^2(t)$;
$\theta^2(t) = \frac{\pi(t - T)^2}{T^2}$.

Fig. 1



$\pi(1-t')^2$	$1-t'$	$t' \equiv \frac{t}{T}$	t	$\sin\{\pi(1-t')^2\}$
0	0.0000	1.0000	T	0
$\pi/16$	0.2500	0.7500	0.7500T	0.19509
$\pi/6$	$1/\sqrt{6} = 0.4082$	0.5918	0.5918T	0.50000
$\pi/4$	0.5000	0.5000	0.5000T	$0.7071 = 1/\sqrt{2}$
$\pi/3$	$1/\sqrt{3} = 0.5773$	0.4227	0.4227T	$0.8660 = \sqrt{3}/2$
$\pi/2$	$1/\sqrt{2} = 0.7071$	0.2929	0.2929T	1.0000
$2\pi/3$	$\sqrt{2}/\sqrt{3} = 0.8165$	0.1835	0.1835T	0.8660
$3\pi/4$	$\sqrt{3}/2 = 0.8660$	0.1340	0.1340T	0.7071
$5\pi/6$	$\sqrt{5}/\sqrt{6} = 0.9129$	0.0871	0.0871T	0.5000
π	1.0000	0.0000	0	0.0000

$$h(t) = \sin\left\{\pi \frac{(T-t)^2}{T^2}\right\} \equiv \sin\{\pi(1-t')^2\} \quad ; \quad t' \equiv \frac{t}{T}$$

Fig. 2

C. Description of the Methods Used in the Evaluation of $G(f, \alpha)$

The evaluation of the integral for $G(f, \alpha)$ in Eq. (3) depends primarily on the evaluation of the related integral

$$D_{on}(f, \alpha) = \int_{s_n}^{t_n} \exp\left[i\left\{\alpha t^2 - 2\pi f t\right\}\right] dt \tag{26}$$

subject to the inequalities stated in Eqs. (6) and (7). For convenience, D_{on} will first be rewritten in another form by completing the square in the exponent and then introducing the substitutions

$$\begin{aligned}
&\xi = \Lambda - \frac{t}{C} \; ; \; \Lambda = 2fC \equiv (\sqrt{2\pi/\alpha})f \; ; \; C = \sqrt{\pi/2\alpha} \; ; \; \alpha = \pi/2C^2 \; ; \\
&v_n = \Lambda - 2B_n \; ; \; 2B_n = \frac{s_n}{C} \; ; \; v_n = (2C)(f - \frac{\alpha s_n}{\pi}) \\
&u_n = \Lambda - 2A_n \; ; \; 2A_n = \frac{t_n}{C} ; \; u_n = (2C)(f - \frac{\alpha t_n}{\pi}) \; .
\end{aligned} \tag{27}$$

After completing the procedure just described, one obtains, in place of $D_{on}(f, \alpha)$, the following integral expression:

$$I_{on}(\Lambda, C) \equiv D_{on}(\frac{\Lambda}{2C}, \frac{\pi}{2C^2}) = C\, e^{-i\frac{\pi}{2}\Lambda^2} J_n(\Lambda, C) \; ; \tag{28}$$

where

$$J_n(\Lambda, C) = \int_{u_n}^{v_n} e^{i\frac{\pi}{2}\xi^2} d\xi \; .$$

I_{on} (and hence D_{on}) can now be evaluated provided the Fresnel-type integral, J_n, in the above equation can be evaluated subject to the inequalities stated in Eqs. (6) and (7). During the course of the work being reported here, it was found that one could evaluate $J_n(\Lambda, C)$ by employing successive integrations by parts, while utilizing certain properties of the curve known as the Cornu Spiral.* For example, the first two integrations by parts are carried out in the following manner:

$$J_n(\Lambda, C) = \int_{u_n}^{v_n} (i\pi\xi)^{-1} \left(i\pi\xi\, e^{i\frac{\pi}{2}\xi^2} d\xi\right) \equiv \int_{u_n}^{v_n} (i\pi\xi)^{-1}\, d\left(e^{i\frac{\pi}{2}\xi^2}\right) ;$$

$$w = (i\pi\xi)^{-1} \ ; \ dw = -(i\pi)^{-1}\xi^{-2} d\xi \ ; \ dv = d\left(e^{i\frac{\pi}{2}\xi^2}\right) ; \ v = e^{i\frac{\pi}{2}\xi^2} ;$$

$$J_n(\Lambda, C) = \left(\frac{1}{i\pi v_n}\right) e^{i\frac{\pi}{2}v_n^2} - \left(\frac{1}{i\pi u_n}\right) e^{i\frac{\pi}{2}u_n^2} + \int_{u_n}^{v_n} \frac{d\left(e^{i\frac{\pi}{2}\xi^2}\right)}{(i\pi)^2\xi^3} ; \qquad (29)$$

$$w = (i\pi)^{-2}\xi^{-3} \ ; \ dw = -(i\pi)^{-2}(3\xi^{-4})d\xi ; \ dv = d\left(e^{i\frac{\pi}{2}\xi^2}\right) ; \ v = e^{i\frac{\pi}{2}\xi^2} ;$$

$$J_n(\Lambda, C) = \left\{\frac{1}{i\pi v_n} + \frac{1}{(i\pi)^2 v_n^3}\right\} e^{i\frac{\pi}{2}v_n^2} - \left\{\frac{1}{i\pi u_n} + \frac{1}{(i\pi)^2 u_n^3}\right\} e^{i\frac{\pi}{2}u_n^2}$$

$$+ \int_{u_n}^{v_n} \frac{(1)(3)}{(i\pi)^3\xi^5}\, d\left(e^{i\frac{\pi}{2}\xi^2}\right) .$$

Continuing in this manner, each time setting dv equal to $d\left(e^{i\frac{\pi}{2}\xi^2}\right)$ and w equal to the rest of the integrand, one obtains, after the $(M+1)^{st}$ integration by parts, the following expansions for the functions J_n and I_{on}:

---

*See Section E.

$$\left.\begin{aligned} J_n(\Lambda, C) &= \sum_{m=0}^{M} \left[V_{mn}(v_n) \exp i(\tfrac{\pi}{2} v_n^2) - U_{mn}(u_n) \exp i(\tfrac{\pi}{2} u_n^2) \right] \\ &\quad + \left\{ R_{Mn}(u_n, v_n) \right\} \\ I_{on}(\Lambda, C) &= \sum_{m=0}^{M} \left[CV_{mn} \exp(i\vartheta_n) - CU_{mn} \exp(i\eta_n) \right] \\ &\quad + \left\{ C \exp(-i\tfrac{\pi}{2} \Lambda^2) R_{Mn} \right\}^{*} \end{aligned} \right\} \quad (30)$$

where

$$\left.\begin{aligned} &V_{mn}(v_n) = \frac{b_m}{v_n^{2m+1}}; \; b_m = -\frac{(-1)(1)(3) --- (2m-1)}{(i\pi)^{m+1}}; \; U_{mn}(u_n) \equiv V_{mn}(u_n) \\ &R_{Mn}(u_n, v_n) = \int_{u_n}^{v_n} \frac{b_{M+1}}{\xi^{2M+3}} d(e^{i\frac{\pi}{2}\xi^2})^{*}; \; b_{M+1} = -\frac{(-1)(1)(3) --- (2M+1)}{(i\pi)^{M+2}} \\ &v_n = (2C)(f - \frac{\alpha s_n}{\pi}); \; u_n = (2C)(f - \frac{\alpha t_n}{\pi}); \; C = \sqrt{\frac{\pi}{2\alpha}}; \\ &\vartheta_n \equiv \frac{\pi}{2}(v_n^2 - \Lambda^2) = \alpha s_n^2 - 2\pi f s_n; \; \eta_n \equiv \frac{\pi}{2}(u_n^2 - \Lambda^2) = \alpha t_n^2 - 2\pi f t_n. \end{aligned} \right\} (30a)$$

The functions I_{on} and J_n in Eqs. (28) and (30) can now be used in the evaluation of certain important related integrals, namely:

$$\mathcal{Q}_{on} = \int_{s_n}^{t_n} a_{on} \exp\left[i\left\{ \alpha t^2 - 2\pi f t \right\} \right] dt = a_{on} D_{on} \quad (31)$$

*The convergence properties of this asymptotic expansion are discussed in the appendix.

$$\mathcal{Q}_{1n} = \int_{s_n}^{t_n} a_{1n} t \exp\left[i\left\{\alpha t^2 - 2\pi f t\right\}\right] dt = a_{1n}\left(\frac{i}{2\pi}\right) \frac{\partial D_{on}}{\partial f}$$

$$\mathcal{Q}_{2n} = \int_{s_n}^{t_n} a_{2n} t^2 \exp\left[i\left\{\alpha t^2 - 2\pi f t\right\}\right] dt = a_{2n}\left(\frac{i}{2\pi}\right)^2 \frac{\partial^2 D_{on}}{\partial f^2} \tag{31}$$

where a_{on}, a_{1n}, and a_{2n} are constants. One or more of these integrals occurs in the evaluation of the spectrum of each of the eight pulses previously listed. For example, all three are applicable when the pulse shape is either parabolinear or parabolic, while only the first two apply in the case of the triangular or the trapezoidal pulse. The above three integrals can be expressed in terms of I_{on} and the first two derivatives of I_{on} with respect to Λ with the aid of the following relationships:

$$\Lambda = 2fC \, ; \; D_{on}(f, \alpha) \equiv D_{on}\left(\frac{\Lambda}{2C}, \frac{\pi}{2C^2}\right) \equiv I_{on}(\Lambda, C) \, ;$$

$$\frac{\partial D_{on}}{\partial f} \equiv \frac{\partial I_{on}}{\partial \Lambda} \frac{\partial \Lambda}{\partial f} \equiv (2C) \frac{\partial I_{on}}{\partial \Lambda} \, ; \; \frac{\partial^2 D_{on}}{\partial f^2} \equiv (2C)^2 \frac{\partial^2 I_{on}}{\partial \Lambda^2} \, . \tag{32}$$

Equation (31) can now be re-written as follows:

$$\mathcal{Q}_{on} = a_{on} I_{on} \, ; \; \mathcal{Q}_{1n} = a_{in}\left(\frac{iC}{\pi}\right) \frac{\partial I_{on}}{\partial \Lambda} \, ; \; \mathcal{Q}_{2n} = a_{2n}\left(\frac{iC}{\pi}\right)^2 \frac{\partial^2 I_{on}}{\partial \Lambda^2} \, . \tag{33}$$

When the differentiations indicated in Eq. (33) have been performed, using the expression for I_{on} given in Eq. (28), one obtains

$$\mathcal{Q}_{on} = (a_{on} C) \left[J_n\right] e^{-i\frac{\pi}{2}\Lambda^2}$$

$$\mathcal{Q}_{1n} = (a_{1n} C^2) \left[\Lambda J_n + (-\frac{1}{i\pi}) J'_n\right] e^{-i\frac{\pi}{2}\Lambda^2} \qquad (34)$$

$$\mathcal{Q}_{2n} = (a_{2n} C^3) \left[(-\frac{1}{i\pi}) J_n + \Lambda^2 J_n + (-\frac{2}{i\pi}) \Lambda J'_n + (\frac{1}{i\pi})^2 J''_n\right] e^{-i\frac{\pi}{2}\Lambda^2}$$

where

$$J'_n \equiv \frac{\partial J_n}{\partial \Lambda} \quad ; \quad J''_n \equiv \frac{\partial^2 J_n}{\partial \Lambda^2}$$

and

$$J'_n e^{-i\frac{\pi}{2}\Lambda^2} \equiv \frac{\partial J_n}{\partial \Lambda} e^{-i\frac{\pi}{2}\Lambda^2} = e^{i\vartheta_n} - e^{i\eta_n}$$

$$J''_n e^{-i\frac{\pi}{2}\Lambda^2} \equiv \frac{\partial^2 J_n}{\partial \Lambda^2} e^{-i\frac{\pi}{2}\Lambda^2} = (i\pi v_n) e^{i\vartheta_n} - (i\pi u_n) e^{i\eta_n} \qquad (35)$$

$$\Lambda = v_n + \frac{s_n}{C} \equiv u_n + \frac{t_n}{C} \; ; \; \frac{\pi}{2}(v_n^2 - \Lambda^2) \equiv \vartheta_n \; ; \; \frac{\pi}{2}(u_n^2 - \Lambda^2) = \eta_n$$

The next step is to replace J_n in **Eq.** (34) by the expansion given in **Eq.** (30) and J'_n and J''_n by the expressions given in **Eq.** (35). The result of making these substitutions and combining terms, while employing $\Lambda = v_n + \frac{s_n}{C}$ in conjunction with terms involving v_n and ϑ_n and $\Lambda = u_n + \frac{t_n}{C}$ in conjunction with terms involving u_n and η_n, is shown in the following equations:

$$\mathcal{Q}_{on} = \left\{ \sum_{m=0}^{M} \left[(a_{on} C) V_{mn} e^{i\vartheta_n} - (a_{on} C) U_{mn} e^{i\eta_n}\right]\right\} + \left\{r_{on}\right\} \qquad (36)$$

$$\mathcal{Q}_{1n} = \left\{ \sum_{m=0}^{M} \left[(a_{1n} s_n C) V_{mn} e^{i\vartheta_n} - (a_{1n} t_n C) U_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \left\{ \sum_{m=1}^{M} \left[(a_{1n} C^2) v_n V_{mn} e^{i\vartheta_n} - (a_{1n} C^2) u_n U_{mn} e^{i\eta_n} \right] \right\} + \left\{ r_{1n} \right\} \quad (37)$$

$$\mathcal{Q}_{2n} = \left\{ \sum_{m=0}^{M} \left[(a_{2n} s_n^2 C) V_{mn} e^{i\vartheta_n} - (a_{2n} t_n^2 C) U_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \left\{ \sum_{m=1}^{M} \left[(2a_{2n} s_n C^2) v_n V_{mn} e^{i\vartheta_n} - (2a_{2n} t_n C^2) u_n U_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \left\{ \sum_{m=2}^{M} \left(\frac{2m-2}{2m-1} \right) \left[(a_{2n} C^3) v_n^2 V_{mn} e^{i\vartheta_n} - (a_{2n} C^3) u_n^2 U_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \left\{ \left(- \frac{a_{2n} C^3}{i\pi} \right) \left(V_{Mn} e^{i\vartheta_n} - U_{Mn} e^{i\eta_n} \right) \right\} + \left\{ r_{2n} \right\} \quad . \quad (38)$$

The remainder terms r_{on}, r_{1n}, and r_{2n} in the above equations are defined as follows:

$$r_{on} = \left(a_{on} C e^{-i\frac{\pi}{2}\Lambda^2} \right) (R_{Mn}) \; ; \; r_{1n} = \left(a_{1n} C^2 e^{-i\frac{\pi}{2}\Lambda^2} \right) (\Lambda) (R_{Mn})$$

$$r_{2n} = \left(a_{2n} C^3 e^{-i\frac{\pi}{2}\Lambda^2} \right) \left(-\frac{1}{i\pi} + \Lambda^2 \right) (R_{Mn}) \; ; \; R_{Mn} = \int_{u_n}^{v_n} \frac{b_{M+1}}{\xi^{2M+3}} \, d\left(e^{i\frac{\pi}{2}\xi^2} \right) ;$$

$$b_{M+1} = - \frac{(-1)(1)(3) \cdots (2M-1)(2M+1)}{(i\pi)^{M+2}} \quad (39)$$

$$v_n = \Lambda - \frac{s_n}{C} = (2C)\left(f - \frac{\alpha s_n}{\pi} \right) ; \; u_n = \Lambda - \frac{t_n}{C} = (2C)\left(f - \frac{\alpha t_n}{\pi} \right)$$

The functions $\mathcal{G}_{on}$, $\mathcal{G}_{1n}$, and $\mathcal{G}_{2n}$ in Eqs. (36), (37), and (38) can now be combined in order to permit the computation of the FM spectra of three basic waveforms, each one defined in the interval $s_n < t < t_n$. The three waveforms referred to above correspond to the following definitions* of the function g(t) appearing in Eq. (1):

$$
\begin{aligned}
g(t) &= h_n(t) = a_{on} = \text{constant} \quad \text{in } s_n < t < t_n \quad \text{(Rectangular Pulse Segment)} \\
g(t) &= h_n(t) = a_{on} + a_{1n} t \quad \text{in } s_n < t < t_n \quad \text{(Linear Pulse Segment)} \\
g(t) &= h_n(t) = a_{on} + a_{1n} t + a_{2n} t^2 \quad \text{in } s_n < t < t_n \quad \text{(Parabolic Pulse Segment)}
\end{aligned}
\qquad (40)
$$

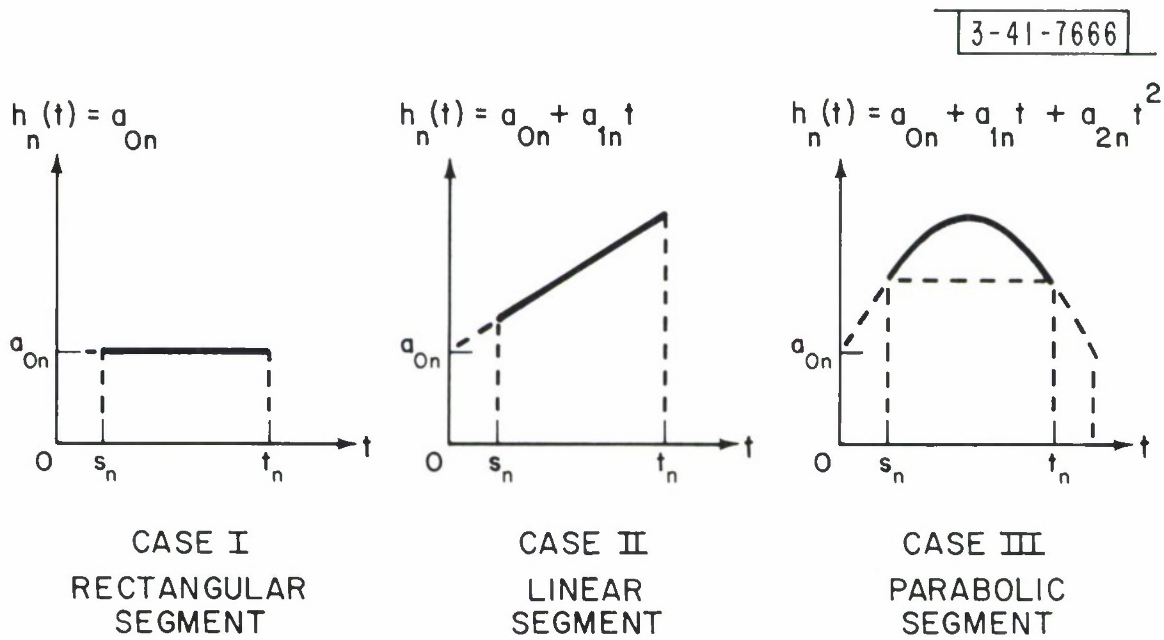


Fig. 3

*See Fig. 3 and Eqs. (40) through (43).

The results can be summarized as follows:

Case I Rectangular Pulse Segment $h_n(t) = a_{on}$ in $s_n < t < t_n$

$$\mathcal{Q}_{on} = \sum_{m=0}^{M} \left\{ h_n(s_n) \left[CV_{mn} e^{i\vartheta_n} \right] - h_n(t_n) \left[CU_{mn} e^{i\eta_n} \right] \right. + \left\{ r_{on} \right\} \quad (41)$$

where

$$h_n(s_n) \equiv h_n(t_n) \equiv a_{on} \quad (41a)$$

Case II Linear Pulse Segment $h_n(t) = a_{on} + a_{1n}t$ in $s_n < t < t_n$

$$\mathcal{Q}_{on} + \mathcal{Q}_{1n} = \sum_{m=0}^{M} \left\{ h_n(s_n) \left[CV_{mn} e^{i\vartheta_n} \right] - h_n(t_n) \left[CU_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \sum_{m=1}^{M} \left\{ h_n'(s_n) \left[C^2 v_n V_{mn} e^{i\vartheta_n} \right] - h_n'(t_n) \left[C^2 u_n U_{mn} e^{i\eta_n} \right] \right\} + \left\{ r_{on} + r_{1n} \right\}$$

(42)

where

$$h_n(s_n) = a_{on} + a_{1n}s_n; \quad h_n(t_n) = a_{on} + a_{1n}t_n; \quad h'(s_s) \equiv h'(t_n) = a_{1n} \quad (42a)$$

Case III Parabolic Pulse Segment $h_n(t) = a_{on} + a_{1n}t + a_{2n}t^2$ in $s_n < t < t_n$

$$\mathcal{Q}_{on} + \mathcal{Q}_{1n} + \mathcal{Q}_{2n} = \sum_{m=0}^{M} \left\{ h_n(s_n) \left[CV_{mn} e^{i\vartheta_n} \right] - h_n(t_n) \left[CU_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \sum_{m=1}^{M} \left\{ h_n'(s_n) \left[C^2 v_n V_{mn} e^{i\vartheta_n} \right] - h_n'(t_n) \left[C^2 u_n U_{mn} e^{i\eta_n} \right] \right\}$$

$$+ \sum_{m=2}^{M} \left\{ \frac{1}{2} \frac{2m-2}{2m-1} \right\} \left\{ h_n''(s_n) \left[C^3 v_n^2 V_{mn} e^{i\vartheta_n} \right] - h_n''(t_n) \left[C^3 u_n^2 U_{mn} e^{i\eta_n} \right] \right\}$$

$$- \left\{ \frac{h_n''(s_n)}{2\pi i} \left[C^3 V_{Mn} e^{i\vartheta_n} \right] - \frac{h_n''(t_n)}{2\pi i} \left[C^3 U_{Mn} e^{i\eta_n} \right] \right\} + \left\{ r_{on} + r_{1n} + r_{2n} \right\}$$

(43)

where

$$h_n(s_n) = a_{on} + a_{1n}s_n + a_{2n}s_n^2 \; ; \; h_n(t_n) = a_{on} + a_{1n}t_n + a_{2n}t^2$$

(43a)

$$h'_n(s_n) = a_{1n} + 2a_{2n}s_n \; ; \; h'_n(t_n) = a_{1n} + 2a_{2n}t_n ; \; h''_n(a_n) \equiv h''_n(t_n) = 2a_{2n}$$

D. Outline of the Problem of Evaluating the FM Spectrum, $G(f, \alpha)$, of a Pulse Describable as a Piecewise Linear Function of Time in the Interval $0 \le t \le T$

To illustrate the use of the formulas in Eqs. (41) through (43a) in calculating the FM spectrum of a useful function, the problem of calculating the spectrum of a pulse which can be approximated by a piecewise linear function will now be considered briefly. In this problem, using the definition of h(t) for the piecewise linear case, as given in Eqs. (8) and (9), one is faced with the evaluation of the following integral:

$$\left.\begin{aligned} G(f,\alpha) &= \int_0^T h(t)\, e^{i\left[\alpha t^2 - 2\pi f t\right]}\, dt \; ; \\ h(t) &= h_n(t) \;\text{ in }\; (n-1)\left(\frac{T}{N}\right) \le t \le \frac{nT}{N} \; ; \; h(t) = a_{on} + a_{1n} t \; , \end{aligned}\right\} \qquad (44)$$

which is a special case of the general integral defined in Eqs. (1), (2), and (3) of this report. In performing the required evaluation of the integral in Eq. (44), Case II, as described in Eqs. (42) and (42a), is the case which applies. Equation (44) can thus be rewritten in the following manner:

$$\begin{aligned} G(f,\alpha) &= \sum_{n=1}^{N} \left\{ \int_{s_n}^{t_n} a_{on}\, e^{i\left[\alpha t^2 - 2\pi f t\right]}\, dt + \int_{s_n}^{t_n} a_{1n}\, t\, e^{i\left[\alpha t^2 - 2\pi f t\right]}\, dt \right\} \\ &\equiv \sum_{n=1}^{N} \left\{ \mathcal{Q}_{on} + \mathcal{Q}_{1n} \right\} \end{aligned} \qquad (45)$$

where

$$s_n = (n-1)\left(\frac{T}{N}\right) \equiv \frac{nT}{N} - \frac{T}{N} \; ; \; t_n = n\left(\frac{T}{N}\right) \equiv \frac{nT}{N} \; .$$

A formal expression for the spectrum described in Eq. (44) can now be obtained by substituting in Eq. (45) the asymptotic expansion of the function $\mathcal{Q}_{on} + \mathcal{Q}_{1n}$ given in Eqs. (42) and (42a). Once again, subject to the assumption stated in Eq. (7), a suitable approximation for $G(f, \alpha)$ can be obtained by retaining only the first non-vanishing term of each of the expansions given in Eq. (42). The addition of terms such as these for all of the N intervals into which the interval $0 \leq t \leq T$ has been divided of course entails additional mathematical complications. However, these are not too severe, especially, as is often the case, when the pulse shape $h(t)$ can be adequately approximated by a piecewise linear function for which the number, N, of sub-intervals is between 5 and 10. For values of N smaller than about 5, the computations required are relatively simple. In particular, Case II was employed with a minimum of difficulty in the calculation of the spectrum of the triangular pulse, the results of the calculation having been presented earlier in Eq. (13). In this case only two sub-intervals were used (i. e., $N = 2$). Similarly, the spectrum of the rectangular pulse, as given in Eq. (11), was calculated with the aid of the result given for Case I, with $N = 1$, while the spectra of the trapezoidal, parabolic, and parabolinear pulses, as given in Eqs. (17), (15), and (18), respectively, were calculated using the appropriate combinations of Cases I, II, and III and at most a value of $N = 3$.

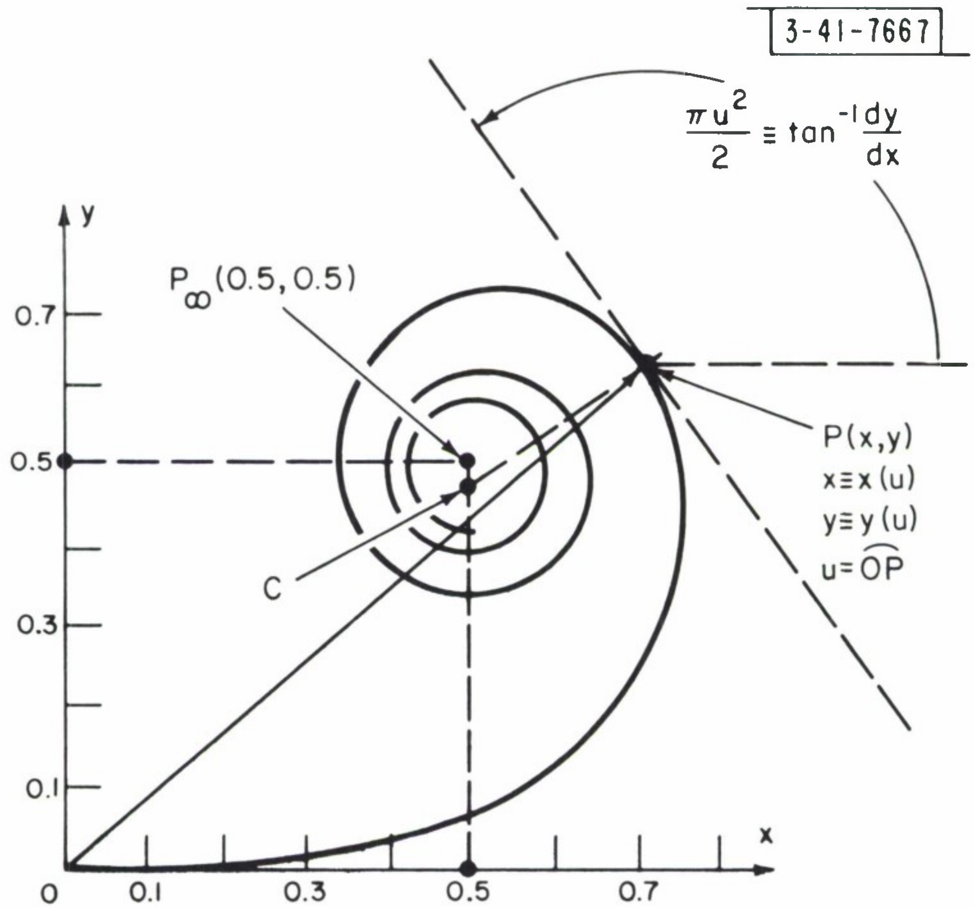
E. Geometrical Interpretation of the First Term of the Expansion of $J_n(\Lambda, C)$

It was stated earlier that certain properties of the curve known as the Cornu Spiral could be utilized in conjunction with the integration-by-parts procedure. While it is not at all necessary to invoke the geometrical characteristics of the Cornu Spiral in this connection, nevertheless by so doing one can provide convenient geometrical interpretations of the various terms in the expansion of $J_n(\Lambda, C)$ given in Eq. (30), and thus provide a much more satisfying analysis than one based on the purely formal integration-by-parts procedure outlined above. In the following, only the first term in the expansion and its geometrical interpretation will be discussed.

The coordinates, x and y, the slope, $\frac{dy}{dx}$, the second derivative, $\frac{d^2y}{dx^2}$, and the radius of curvature, ρ, of the Cornu Spiral at a point $P(x, y)$ are given in the following equation.

$$\left.\begin{aligned}
&x(u) = \int_0^u \cos\frac{\pi}{2}\xi^2\,d\xi \;;\; y(u) = \int_0^u \sin\frac{\pi}{2}\xi^2\,d\xi \;;\; u = \overset{\frown}{OP} \\
&J(u) \equiv x + iy \equiv \int_0^u e^{i\frac{\pi}{2}\xi^2}\,d\xi = \overrightarrow{OP} \quad \text{(see Fig. 4)} \\
&dx = \cos\frac{\pi}{2}u^2\,du \;;\; dy = \sin\frac{\pi}{2}u^2\,du \;; \\
&\frac{dy}{dx} \equiv y' = \tan\frac{\pi}{2}u^2 \;;\; \frac{du}{dx} \equiv u' = \sec\frac{\pi}{2}u^2 \;; \\
&\frac{d^2y}{dx^2} \equiv y'' \equiv u'\frac{dy'}{du} = \pi u \sec^3\frac{\pi}{2}u^2 \;;\; y' = \text{Slope at } P(x, y) \;; \\
&\rho = \text{Radius of Curvature at } P(x, y);\ \rho = \left[1 + y'^2\right]^{3/2} \div y'' = \frac{1}{\pi u}.
\end{aligned}\right\} \quad (46)$$

The function J, as shown in Fig. 4, is the vector $\overrightarrow{OP}$ from the origin 0 to the arbitrary point P on the spiral. Similarly, the function J_n in Eq. (28) is the vector $\overrightarrow{P_nQ_n}$ shown in Fig. 4, or equivalently, the vector difference



$x \equiv x(u) = \int_0^u \cos\frac{\pi}{2}\,\xi^2 d\xi$; $y \equiv y(u) = \int_0^u \sin\frac{\pi}{2}\,\xi^2 d\xi$;

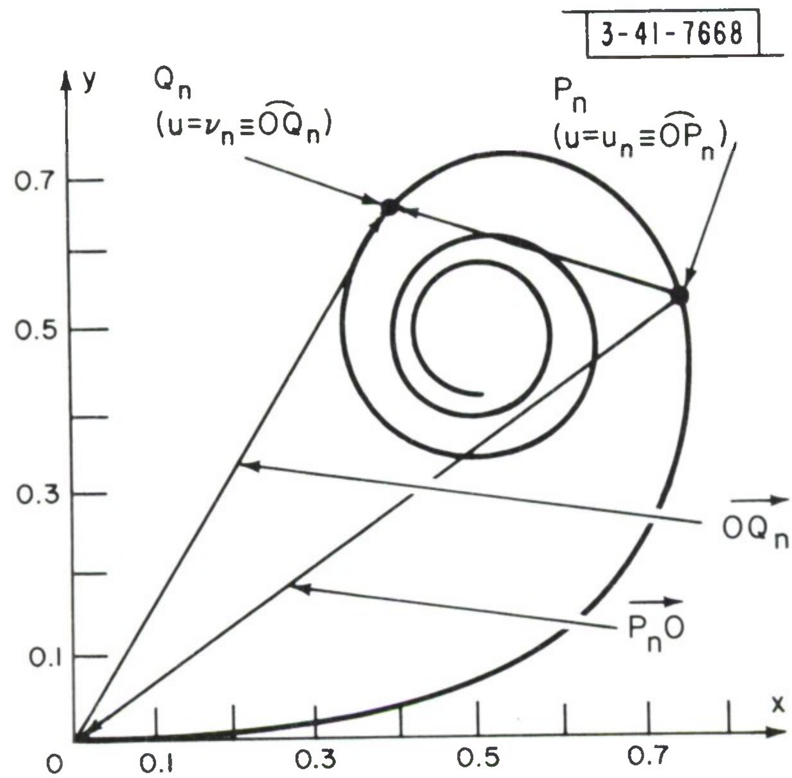
$u = \widehat{OP}$ = arc length at P ; $\frac{dy}{dx} = \tan\frac{\pi u^2}{2}$ = slope at P ;

$J(u) = x + iy \equiv \int_0^u \exp\{i\frac{\pi}{2}\xi^2\}\,d\xi \equiv \overrightarrow{OP}$;

C = center of curvature of spiral at P ;

$\rho \equiv \overline{CP} = \frac{1}{\pi u}$ = radius of curvature at P .

Fig. 4

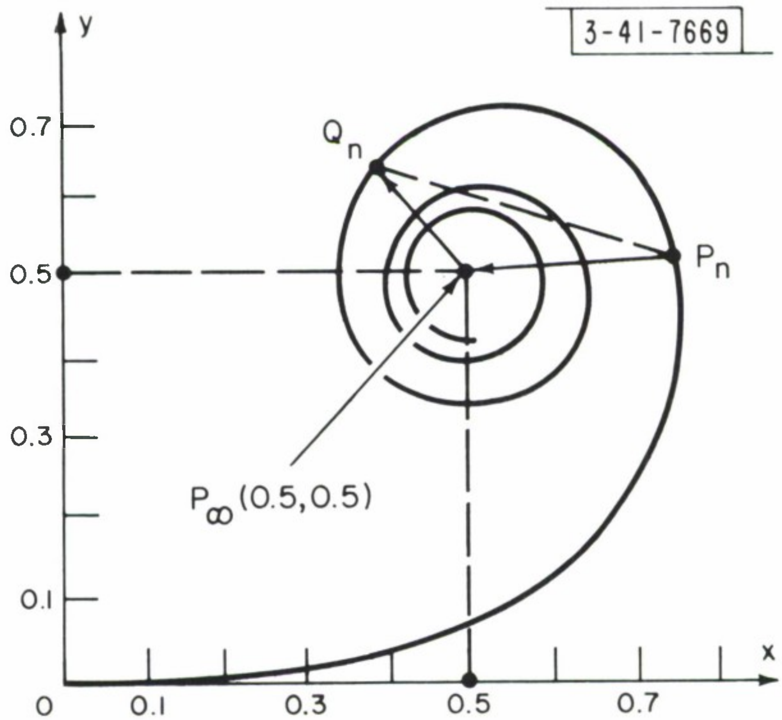


$$J_n(\Lambda, C) = \int_{u_n}^{\nu_n} e^{i\,\pi/2\;\xi^2}\, d\xi$$

$$\equiv \int_0^{\nu_n} e^{i\,\pi/2\;\xi^2}\, d\xi - \int_0^{u_n} e^{i\,\pi/2\;\xi^2}\, d\xi$$

$$\equiv \overrightarrow{OQ_n} - \overrightarrow{OP_n} \equiv \overrightarrow{P_nQ_n} \quad .$$

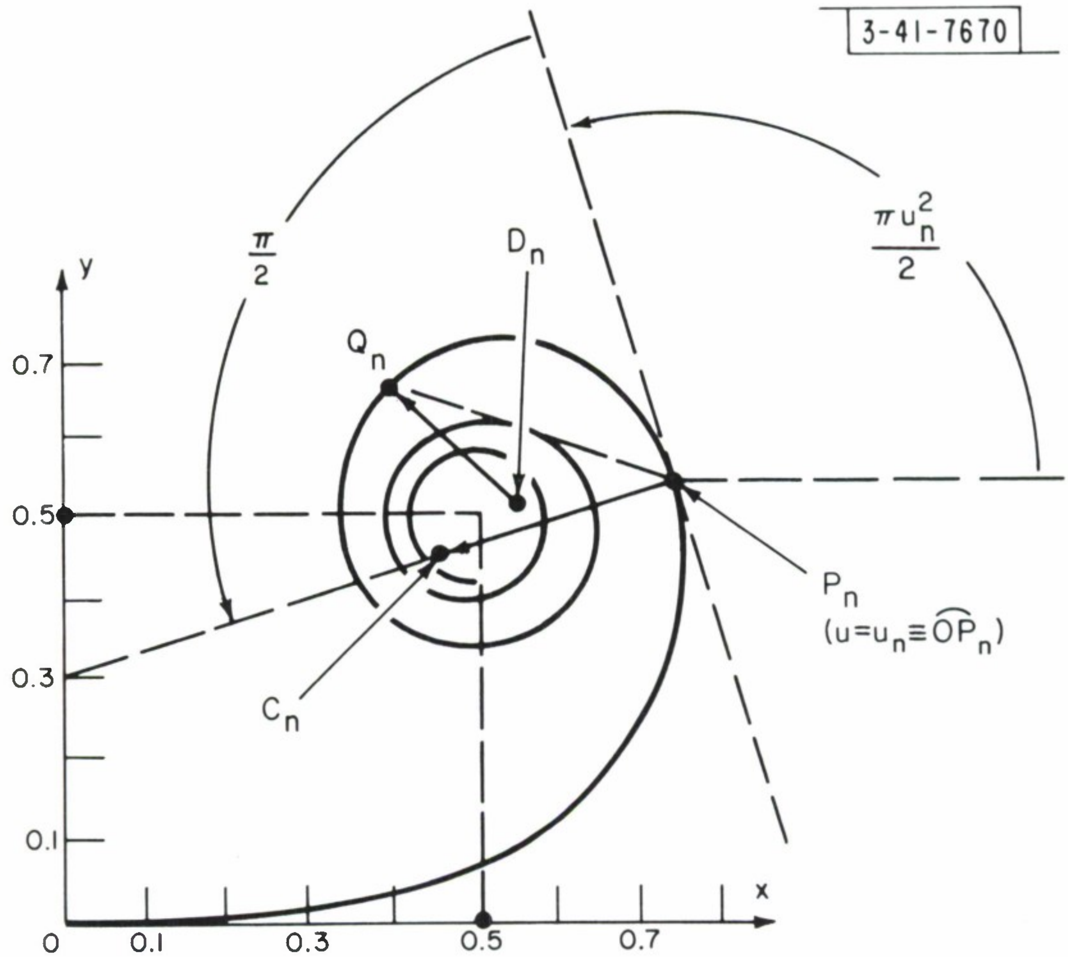
Fig. 5



$$J_n(\Lambda, C) = \overrightarrow{P_n Q_n} \equiv \overrightarrow{P_n P_\infty} + \overrightarrow{P_\infty Q_n}$$

$$\equiv \overrightarrow{P_n P_\infty} - \overrightarrow{Q_n P_\infty} \quad .$$

Fig. 6



$$\overrightarrow{P_nP_\infty} \sim \overrightarrow{P_nC_n} \quad ; \quad \overrightarrow{Q_nP_\infty} \sim \overrightarrow{Q_nD_n} \quad ;$$

$$J_n(\Lambda, C) \equiv \overrightarrow{P_nP_\infty} - \overrightarrow{Q_nP_\infty}$$

$$\sim \overrightarrow{P_nC_n} - \overrightarrow{Q_nD_n} \quad .$$

Fig. 7

$\overrightarrow{P_nP_\infty} - \overrightarrow{Q_nP_\infty}$ shown in Fig. 6, where $P_\infty \equiv P_\infty(.5, .5)$ is the limiting point, the point at which the arc length, measured from 0, is infinite. It is clear, after knowing the meaning of $J_n(\Lambda, C)$ and noting the definitions of the upper and lower limits, v_n and u_n, respectively, given in Eq. (27), that when the inequality in Eq. (7) is satisfied, both ends of the vector $\overrightarrow{P_nQ_n}$ (i.e., the function $J_n(\Lambda, C)$) lie far within the wound-up portion of the spiral. In this portion of the spiral, the center of curvature, C, of the spiral and the limiting point, P_∞, are very nearly coincident, so nearly so in fact, that in a great many applications (but not in the present one) the following is an excellent approximation for the integral $J_n(\Lambda, C)$. (See also Fig. 7.)

$$J_n(\Lambda, C) = \int_{u_n}^{v_n} e^{i\frac{\pi}{2}\xi^2}\, d\xi = \overrightarrow{P_nP_\infty} - \overrightarrow{Q_nP_\infty}$$

$$J_n(\Lambda, C) \sim \overrightarrow{P_nC_n} - \overrightarrow{Q_nD_n} = \frac{\exp\left[i\left\{\frac{\pi}{2} + \frac{\pi}{2}u_n^2\right\}\right]}{\pi u_n} - \frac{\exp\left[i\left\{\frac{\pi}{2} + \frac{\pi}{2}v_n^2\right\}\right]}{\pi v_n} \tag{47}$$

where $\frac{1}{\pi u_n}$ and $\frac{1}{\pi v_n}$ are the radii of curvature at P_n and Q_n respectively. Equation (47) was obtained by first noting that the lengths of the vectors $\overrightarrow{P_nC_n}$ and $\overrightarrow{Q_nD_n}$ are simply the radii of curvature $\frac{1}{\pi u_n}$ and $\frac{1}{\pi v_n}$, respectively, at the two points P_n and Q_n. Next, it is noted that the phase angles of the two vectors are $\frac{\pi}{2} + \frac{\pi}{2}u_n^2$ and $\frac{\pi}{2} + \frac{\pi}{2}v_n^2$, respectively. For example, the term $\frac{\pi}{2}u_n^2$ is simply the <u>angle of inclination</u> of the spiral at P_n, as given in general in Eq. (46). The remaining angle, $\frac{\pi}{2}$, appears because the spiral is being approximated at P_n by a circle centered at C_n, and thus $\overrightarrow{P_nC_n}$ is perpendicular to the spiral. The above ideas are illustrated schematically in Fig. 7.

Upon comparing Eq. (47) with the third line of Eq. (29), one sees that the approximation given in Eq. (47) is none other than the FIRST TERM in the expansion $J_n(\Lambda, C)$. Now, the locus of the center of curvature of the spiral is called the EVOLUTE OF THE SPIRAL. One can easily show that the second term in the expansion (terms in $\frac{1}{u_n^3}$ and $\frac{1}{v_n^3}$) is related to a pair of vectors pointing from C_n to the center of curvature of the EVOLUTE at C_n and from D_n to the center of curvature of the EVOLUTE at D_n. Continuing in this manner, one can give geometrical interpretations to all of the remaining terms, although the geometry involved becomes increasingly complicated with each succeeding term.

APPENDIX A

Some Aspects of the Convergence Properties of the Asymptotic Expansions of the Fresnel Integrals $J_n(\Lambda, C)$ and $I_{on}(\Lambda, C)$

In Section C an integration-by-parts procedure was used to obtain a purely formal asymptotic expansion of the integrals $J_n(\Lambda, C)$ and $I_{on}(\Lambda, C)$ defined in Eq. (28). The results obtained after one and two integrations were given in Eq. (29). The general result of employing M+1 (recall the index m runs from 0 to M) integrations was shown in Eqs. (30) and (30a). However, nothing was said at the time concerning the convergence properties of the expansions obtained in this purely formal manner. In particular, nothing was said concerning the magnitude of the error incurred by neglecting the remainder term R_{Mn} in Eq. (30) and using the M+1 terms in the expansion to approximate the integral.

The purpose of this appendix is to obtain an <u>upper bound</u> on the magnitude of the error introduced by discarding all but a finite number of terms in the asymptotic expansion of the function J_n (and hence I_{on}) given in Eq. (30). The decidedly more difficult problem of estimating a <u>lower bound</u> on this error will not be treated here, but it certainly merits attention at some later date. To aid in the estimation of the upper bound, the expression for $J_n(\Lambda, C)$ given in Eq. (30) will now be rewritten in the following form:

$$J_n(\Lambda, C) \equiv \int_{u_n}^{v_n} e^{i\frac{\pi}{2}\xi^2}\, d\xi = \left\{ \sum_{m=0}^{M} T_{mn}(u_n, v_n) \right\} + \left\{ R_{Mn}(u_n, v_n) \right\}, \quad \text{(A-1)}$$

where

$$\left.\begin{aligned} T_{mn} &= V_{mn}(v_n)\, e^{i\frac{\pi}{2}v_n^2} - U_{mn}(u_n)\, e^{i\frac{\pi}{2}u_n^2}\ ; \quad R_{Mn} = \int_{u_n}^{v_n} \frac{b_{M+1}}{\xi^{2M+3}}\, d\left(e^{i\frac{\pi}{2}\xi^2}\right); \\ V_{mn} &= \frac{b_m}{v_n^{2m+1}}\ ; \quad U_{mn} = \frac{b_m}{u_n^{2m+1}}\ ; \quad b_m = -\frac{(-1)(1)(3)\ \ldots\ldots\ (2m-1)}{(i\pi)^{m+1}} \end{aligned}\right\}$$

(A-2)

$$\left.\begin{aligned} T_{Mn} &= V_{Mn}\, e^{i\frac{\pi}{2} v_n^2} - U_{Mn}\, e^{i\frac{\pi}{2} u_n^2} \;;\; b_{M+1} \equiv \frac{(2M+1) b_M}{i\pi} \;; \\ v_n &= (2C)(f - \frac{\alpha s_n}{\pi}) ;\; u_n = (2C)(f - \frac{\alpha t_n}{\pi}) \;;\; 2C = \sqrt{\frac{2\pi}{\alpha}} \;. \end{aligned}\right\} \qquad \text{(A-2)}$$

In what follows use will also be made of the following important inequalities:

$$s_n < t_n \;;\; v_n > u_n \;;\; (\frac{1}{u_n^{2M+1}}) > (\frac{1}{v_n^{2M+1}}) \;;\; |U_{Mn}| > |V_{Mn}| , \qquad \text{(A-3)}$$

which follow immediately from the definitions of u_n and v_n given above and in Eq. (27) and from the way in which the limits of integration s_n and t_n appear in the definition of the basic integral $D_{on}(f, \alpha)$ given earlier in Eq. (26):

$$D_{on}(f, \alpha) = \int_{s_n}^{t_n} \exp\left[i\left\{\alpha t^2 - 2\pi f t\right\}\right] dt \quad . \qquad (26)$$

A comparison will now be made of the magnitude of T_{Mn}, the $(M+1)^{st}$ term in the asymptotic expansion of J_n, and the magnitude of R_{Mn}, the remainder integral one obtains after performing M+1 integrations by parts. The term T_{Mn} can be rewritten as follows:

$$T_{Mn} = (-b_M\, e^{i\frac{\pi}{2} u_n^2}) \left[(\frac{1}{u_n^{2M+1}}) - (\frac{1}{v_n^{2M+1}})\, e^{i\frac{\pi}{2}(v_n^2 - u_n^2)} \right] . \qquad \text{(A-4)}$$

The magnitude, that is, the absolute value, of this complex vector function is seen immediately to be:

$$|T_{Mn}| = |b_M| \left| \left[(\frac{1}{u_n^{2M+1}}) - (\frac{1}{v_n^{2M+1}})\, e^{i\frac{\pi}{2}(v_n^2 - u_n^2)} \right] \right| , \qquad \text{(A-5)}$$

where use was made of the inequalities stated in Eq. (A-3). Required also are the maximum and minimum values of the function $|T_{Mn}|$. They are:

$$|T_{Mn}|_{MAX} = \frac{|b_M|}{u_n^{2M+1}} + \frac{|b_M|}{v_n^{2M+1}} , \tag{A-6}$$

obtained by setting $\frac{\pi}{2}(v_n^2 - u_n^2)$ in Eq. (A-5) equal to $(2j+1)\pi$ $(j = 0, \pm 1, \pm 2, \ldots)$ and

$$|T_{Mn}|_{MIN} = \frac{|b_M|}{u_n^{2M+1}} - \frac{|b_M|}{v_n^{2M+1}} , \tag{A-7}$$

obtained by setting $\frac{\pi}{2}(v_n^2 - u_n^2) = 2j\pi$ $(j = 0, \pm 1, \pm 2, \ldots)$.

The magnitude of the remainder integral R_{Mn} can best be examined by rewriting this integral, as defined in Eqs. (A-2) and (30a), in the following form:

$$R_{Mn} = \int_{u_n}^{v_n} \frac{b_{M+1}}{\xi^{2M+3}} \, d\left(e^{i\frac{\pi}{2}\xi^2}\right) \equiv \int_{u_n}^{v_n} \frac{(2M+1)\, b_M}{\xi^{2M+2}} \, e^{i\frac{\pi}{2}\xi^2} \, d\xi . \tag{A-8}$$

Because of the difficulties involved in integrating the above expression, a closed-form expression for the magnitude of R_{Mn} is unattainable, although approximate integration techniques can be invoked if necessary to give a result close to the true value. No attempt in this direction will be made here, however. Instead, an estimate of an <u>upper bound</u> (that is, a <u>maximum</u> value) on the magnitude of R_{Mn} will be deemed sufficient to meet the needs of the present analysis. This upper bound can be easily obtained if it is observed that for the range of values of u_n and v_n under consideration (as determined by the definitions of u_n and v_n in Eq. (27) and by the fundamental inequalities stated in Eqs. (6) and (7)) the phase of the complex exponential term $e^{i\frac{\pi}{2}\xi^2}$ in Eq. (A-8) is an extremely rapidly oscillating function of ξ and thus in the region of integration, $u_n < \xi < v_n$, there occurs a considerable amount of destructive interference between the incremental complex vectors

$\frac{(2M+1)\, b_M}{\xi^{2M+2}}\, e^{i\frac{\pi}{2}\xi^2}\, d\xi$ involved in the integration process. On the basis of the above ideas, one can immediately deduce the following very strong inequality:

$$|R_{Mn}| \ll \int_{u_n}^{v_n} \left| \frac{(2M+1)\, b_M}{\xi^{2M+2}} \right| d\xi \equiv \frac{|b_M|}{u_n^{2M+1}} - \frac{|b_M|}{v_n^{2M+1}} \; . \qquad \text{(A-9)}$$

Combining the results presented in Eqs. (A-6), (A-7), and (A-9), one now obtains the following relationship between the various magnitudes:

$$|R_{Mn}| \ll \left(\frac{|b_M|}{u_n^{2M+1}} - \frac{|b_M|}{v_n^{2M+1}} \right) \equiv |T_{Mn}|_{MIN} < |T_{Mn}|_{MAX} \; . \qquad \text{(A-10)}$$

From this equation one obtains immediately the required relationship defining an <u>upper bound</u> on the magnitude of the error introduced by neglecting the remainder term R_{Mn} appearing in the asymptotic expansion of the function J_n, namely:

$$|R_{Mn}| \ll |T_{Mn}| \; . \qquad \text{(A-11)}$$

Equation (A-11) shows that the magnitude of the remainder term is always much less than the magnitude of the $(M+1)^{st}$ term in the series, and a closer examination reveals that, for a <u>fixed</u> value of M, the magnitude of the remainder (and hence of the error) grows continually smaller with increasing u_n and v_n. It is evident, then, that, by terminating the series at a term whose absolute value corresponds nearly to the maximum error tolerable in the calculation of the function J_n, one can establish an <u>upper bound</u> on the error incurred by neglecting the rest of the terms. The same considerations of course also apply to the calculation of the related function I_{on} in Eq. (30). Finally, these ideas can be extended relatively easily to the discussion of the convergence properties of more general asymptotic expansions such as those presented in Eqs. (42) and (43), and, in particular, expansions of integrals of the following general type:

$$G_n(f, \alpha) = \int_{s_n}^{t_n} (a_{on} + a_{1n} t + a_{2n} t^2) \, e^{i\left[\alpha t^2 - 2\pi f t\right]} dt \; . \tag{A-12}$$

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